AUTOMORPHISMS OF CURVES FIXING THE ORDER TWO POINTS OF THE JACOBIAN

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ABSTRACT. Let X be an irreducible smooth projective curve, of genus at least two, defined over an algebraically closed field of characteristic different from two. If X admits a nontrivial automorphism σ that fixes pointwise all the order two points of $\mathrm{Pic}^0(X)$, then we prove that X is hyperelliptic with σ being the unique hyperelliptic involution. As a corollary, if a nontrivial automorphisms σ' of X fixes pointwise all the theta characteristics on X, then X is hyperelliptic with σ' being its hyperelliptic involution.

1. INTRODUCTION

Let Y be a compact connected Riemann surface of genus at least two. Assume that there is a nontrivial holomorphic automorphism

$$\sigma_0 : Y \longrightarrow Y$$

satisfying the condition that for each holomorphic line bundle ξ over Y with $\xi^{\otimes 2}$ trivializable, the pull back $\sigma_0^*\xi$ is holomorphically isomorphic to ξ. In [2] it was shown that Y must be hyperelliptic and σ_0 is the unique hyperelliptic involution (see [2, p. 494, Theorem 1.1]).

We recall that a theta characteristic on Y is a holomorphic line bundle θ such that $\theta^{\otimes 2}$ is holomorphically isomorphic to the homomorphic cotangent bundle K_Y. The group of order two line bundles on Y acts freely transitively on the set of all theta characteristics on Y. From this it follows immediately that if an automorphism of Y fixes pointwise all the theta characteristics, then it also fixes pointwise all the order two line bundles on Y. Therefore, if Y admits a nontrivial automorphism σ'_0 that fixes pointwise all the theta characteristics on Y, then Y is hyperelliptic and σ'_0 is its unique hyperelliptic involution.

The proof of Theorem 1.1 in [2] is topological. Here we investigate the corresponding algebraic geometric set–up, where the topological proof of Theorem 1.1 in [2] is no longer valid.

Let X be an irreducible smooth projective curve defined over an algebraically closed field k. We will assume that $\mathrm{genus}(X) > 1$ and $\mathrm{char}(k) \neq 2$. We prove the following:

Theorem 1.1. *Let*

$$\sigma : X \longrightarrow X$$

be a nontrivial automorphism that fixes pointwise all the theta characteristics on X. Then X is hyperelliptic with σ being its unique hyperelliptic involution.

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This theorem is proved by showing that if

$$\sigma' : X \longrightarrow X$$

is a nontrivial automorphism of X that fixes pointwise all the order two points in $\mathrm{Pic}^0(X)$, then X is hyperelliptic with σ' being its unique hyperelliptic involution. (See Lemma 3.1.)

It should be pointed out that Theorem 1.1 is not valid if the assumption that the field k is algebraically closed is removed. There exists a geometrically irreducible smooth projective real algebraic curve Y of genus $g \geq 2$ which admits a nontrivial involution σ that fixes pointwise all the real points $\xi \in \mathrm{Pic}^{g-1}(Y)$ with $\xi^{\otimes 2} = K_Y$, and $\mathrm{genus}(Y/\langle\sigma\rangle) \neq 0$. (The details are in [1].)

2. Automorphisms of polarized abelian varieties

Let k be an algebraically closed field whose characteristic is different from two. Let A be an abelian variety defined over k and L an ample line bundle over A. For any positive integer n, let

$$A_n \subset A \tag{1}$$

be the scheme–theoretic kernel of the endomorphism $A \longrightarrow A$ defined by $x \longmapsto nx$.

Proposition 2.1. *Let*

$$\tau : A \longrightarrow A$$

be a nontrivial automorphism such that $\tau^*L = L \bigotimes L_0$ *for some* $L_0 \in \mathrm{Pic}^0(A)$, *and the restriction of* τ *to the subscheme* A_{n_0} *(see Eq. (1)) is the identity map for some* $n_0 \geq 2$. *Define the two endomorphisms*

$$f_{\pm} := \mathrm{Id}_A \pm \tau : A \longrightarrow A\,.$$

Let A_+ *(respectively,* A_-*) be the image of* f_+ *(respectively,* f_-*). Then*

(1) $n_0 = 2$.
(2) $\tau^2 = \tau \circ \tau$ *is the identity automorphism of* A.
(3) *The natural homomorphism*

$$\beta : A_+ \times A_- \longrightarrow A \tag{2}$$

defined by the inclusions of A_+ *and* A_- *in* A *is an isomorphism.*

(4) *The pull back* β^*L *is of the form* $p_+^*L_+ \bigotimes p_-^*L_-$, *where* p_+ *(respectively,* p_-*) is the projection of* $A_+ \times A_-$ *to* A_+ *(respectively,* A_-*).*

Proof. A proof of statement (1) is given in [4, p. 207, Thoerem 5]. See [3, p. 120, Corollary 1.10] for a proof under the assumption that k is the field of complex numbers.

To prove statement (2), we will show that the restriction of τ^2 to A_4 is the identity map. Take any point $x \in A_4$. Then $\tau(2x) = 2x$ because $2x \in A_2$. Hence $\tau(x) = x' + x$ for some $x' \in A_2$. Thus

$$\tau(\tau(x)) = \tau(x' + x) = \tau(x') + \tau(x) = x' + (x' + x) = x\,.$$

Consequently, the restriction of τ^2 to A_4 is the identity map. Now statement (2) follows from statement (1).

To prove statement (3), consider the composition homomorphism

$$A \stackrel{f_+\times f_-}{\longrightarrow} A_+\times A_- \stackrel{\beta}{\longrightarrow} A\,,$$

where β is the homomorphism in Eq. (2). It coincides with the endomorphism of A defined by $x \longmapsto 2x$. We also note that $A_2 \subset \mathrm{kernel}(f_+\times f_-)$. Hence

$$\mathrm{kernel}(\beta\circ(f_+\times f_-)) \subset \mathrm{kernel}(f_+\times f_-)\,. \tag{3}$$

Since $\tau^2 = \mathrm{Id}_A$, the composition $f_+\circ f_-$ is the zero homomorphism. Hence $\dim(A_+\times A_-) \leq \dim A$. Now From Eq. (3) it follows that β is an isomorphism.

To prove statement (4), let

$$\phi_{\beta^*L}\,:\, A_+\times A_- \longrightarrow \mathrm{Pic}^0(A_+\times A_-) \,=\, \mathrm{Pic}^0(A_+)\times\mathrm{Pic}^0(A_-)$$

be the homomorphism that sends any k–rational point $x\in A_+\times A_-$ to the line bundle $(t_x^*\beta^*L)\bigotimes\beta^*L^*$, where t_x is the translation map of $A_+\times A_-$ defined by $y\longmapsto y+x$; see [4, p. 131, Corollary 5] for a precise definition of the morphism ϕ_{β^*L}. Let

$$\tau' \,:=\, \mathrm{Id}_{A_+}\times(-\mathrm{Id}_{A_-})$$

be the automorphism of $A_+\times A_-$. We note that the isomorphism β in Eq. (2) takes τ to τ'.

Let

$$\widehat{\tau}' \,:=\, \mathrm{Id}_{\mathrm{Pic}^0(A_+)}\times(-\mathrm{Id}_{\mathrm{Pic}^0(A_-)})$$

be the automorphism of $\mathrm{Pic}^0(A_+)\times\mathrm{Pic}^0(A_-) \,=\, \mathrm{Pic}^0(A_+\times A_-)$. Since $\tau^*L \,=\, L\bigotimes L_0$ for some $L_0\in\mathrm{Pic}^0(A)$, the following diagram is commutative

$$\begin{array}{ccc}
A_+\times A_- & \stackrel{\phi_{\beta^*L}}{\longrightarrow} & \mathrm{Pic}^0(A_+)\times\mathrm{Pic}^0(A_-)\\
\Big\downarrow\tau' & & \Big\downarrow\widehat{\tau}'\\
A_+\times A_- & \stackrel{\phi_{\beta^*L}}{\longrightarrow} & \mathrm{Pic}^0(A_+)\times\mathrm{Pic}^0(A_-)
\end{array}$$

Therefore, the homomorphism ϕ_{β^*L} takes the subgroup A_+ (respectively, A_-) of $A_+\times A_-$ to the subgroup $\mathrm{Pic}^0(A_+)$ (respectively, $\mathrm{Pic}^0(A_-)$) of $\mathrm{Pic}^0(A_+)\times\mathrm{Pic}^0(A_-)$. Now from the injectivity of the homomorphism

$$\mathrm{NS}(A_+\times A_-) \longrightarrow \mathrm{Hom}(A_+\times A_-\,,\mathrm{Pic}^0(A_+)\times\mathrm{Pic}^0(A_-))$$

defined by $\xi\longmapsto\phi_\xi$ it follows immediately that the Néron–Severi class of β^*L coincides with that of some line bundle of the form $p_+^*L_+\bigotimes p_-^*L_-$ (see [4, p. 178] for the injectivity of the above homomorphism). Therefore, statement (4) follows using the fact that $\mathrm{Pic}^0(A_+)\times\mathrm{Pic}^0(A_-) \,=\, \mathrm{Pic}^0(A_+\times A_-)$. This completes the proof of the proposition. $\square$

3. Automorphisms and theta characteristics

Let X be an irreducible smooth projective curve, of genus at least two, defined over the field k.

Lemma 3.1. *Let*
$$\sigma : X \longrightarrow X$$
be a nontrivial automorphism of X that fixes pointwise all the order two points $\mathrm{Pic}^0(X)_2 \subset \mathrm{Pic}^0(X)$*. then X is hyperelliptic with σ being its unique hyperelliptic involution.*

Proof. Let $\mathrm{Pic}^d(X)$ denote the moduli space of line bundles over X of degree d. Let g denote the genus of X. On $\mathrm{Pic}^{g-1}(X)$, we have the theta divisor Θ given by the locus of the line bundles admitting nontrivial sections. Fix a k–rational point $x_0 \in X$. Let L be the pull back of the line bundle $\mathcal{O}_{\mathrm{Pic}^{g-1}(X)}(\Theta)$ by the morphism $\mathrm{Pic}^0(X) \longrightarrow \mathrm{Pic}^{g-1}(X)$ that sends any ζ to $\zeta \bigotimes \mathcal{O}_X((g-1)x_0)$.

Let $\tau : \mathrm{Pic}^0(X) \longrightarrow \mathrm{Pic}^0(X)$ be the automorphism defined by $\zeta \longmapsto \sigma^*\zeta$. This τ satisfies the conditions in Proposition 2.1. Hence τ is an involution (see Proposition 2.1(2)). This implies that σ is an involution.

A hyperelliptic smooth projective curve Y of genus at least two admits a unique involution σ_Y such that $\mathrm{genus}(Y/\langle\sigma_Y\rangle) = 0$. Therefore, to complete the proof of the lemma it suffices to show that $\mathrm{genus}(X/\langle\sigma\rangle) = 0$. We note that the theta divisor Θ on $\mathrm{Pic}^{g-1}(X)$ is irreducible. Indeed, it is the image of $\mathrm{Sym}^{g-1}(X)$ by the obvious map. Also, $h^0(\mathcal{O}_{\mathrm{Pic}^{g-1}(X)}(\Theta)) = 1$ because Θ defines a principal polarization.

On the other hand, any ample hypersurface of the form $(A_+ \times D_-)\bigcup(D_+ \times A_-)$ on $A_+ \times A_-$ is never irreducible unless at least one of A_+ and A_- is a point; here D_+ (respectively, D_-) is a hypersurface on A_+ (respectively, A_-). Therefore, from statement (4) of Proposition 2.1 and the irreducibility of Θ we conclude that either $\dim A_+ = 0$ or $\dim A_- = 0$. But $\dim A_- = \mathrm{genus}(X) - \mathrm{genus}(X/\langle\sigma\rangle)$, and $\dim A_+ = \mathrm{genus}(X/\langle\sigma\rangle)$. Since $\mathrm{genus}(X) > \mathrm{genus}(X/\langle\sigma\rangle)$, we now conclude that $\mathrm{genus}(X/\langle\sigma\rangle) = 0$. This completes the proof of the lemma. □

A line bundle θ is called a *theta characteristic* of X if $\theta^{\otimes 2}$ is isomorphic to the canonical line bundle K_X of X. The space of theta characteristics on X is a principal homogeneous space for $\mathrm{Pic}^0(X)_2$. Therefore, if an automorphism σ of X fixes pointwise all the theta characteristics on X, then σ fixes $\mathrm{Pic}^0(X)_2$ pointwise. Consequently, the following theorem is deduced from Lemma 3.1.

Theorem 3.2. *Let $\sigma : X \longrightarrow X$ be a nontrivial automorphism that fixes pointwise all the theta characteristics on X. Then X is hyperelliptic with σ being its unique hyperelliptic involution.*

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